NON-SINGULAR SOLUTIONS OF NORMALIZED RICCI FLOW ON NONCOMPACT MANIFOLDS OF FINITE VOLUME

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Abstract. The main result of this paper shows that, if $g(t)$ is a complete non-singular solution of the normalized Ricci flow on a noncompact 4-manifold M of finite volume, then the Euler characteristic number $\chi(M) \geq 0$. Moreover, $\chi(M) \neq 0$, there exist a sequence times $t_k \to \infty$, a double sequence of points $\{p_{k,l}\}_{l=1}^N$ and domains $\{U_{k,l}\}_{l=1}^N$ with $p_{k,l} \in U_{k,l}$ satisfying the followings:

(i) $\mathrm{dist}_{g(t_k)}(p_{k,l_1}, p_{k,l_2}) \to \infty$ as $k \to \infty$, for any fixed $l_1 \neq l_2$;

(ii) for each l, $(U_{k,l}, g(t_k), p_{k,l})$ converges in the C^∞_{loc} sense to a complete negative Einstein manifold $(M_{\infty,l}, g_{\infty,l}, p_{\infty,l})$ when $k \to \infty$;

(iii) $\mathrm{Vol}_{g(t_k)}(M \setminus \bigcup_{l=1}^N U_{k,l}) \to 0$ as $k \to \infty$.

1. Introduction

In his pioneer paper [14], Hamilton considered one special class of Ricci flow solutions on closed three manifolds: non-singular solutions. Hamilton showed that such solutions provides an example of Thurston's geometric decomposition. More precisely, as the time tends to infinity, the manifolds admit thick-thin decomposition, where the thick parts converge to hyperbolic spaces, while the thin parts collapse. In particular, closed 3-manifolds admitting non-singular solutions are geometrizable.

The normalized Ricci flow on a given manifold M is a smooth family of metrics $g(t), t \in [0, T)$, satisfying the evolution equation

$$\frac{\partial}{\partial t} g = -2Ric + \frac{2r}{n} g \tag{1}$$

where Ric denotes the Ricci tensor of g and $r = \frac{\int_M Rdv}{\mathrm{Vol}(g)}$ denotes the average scalar curvature of g. The flow requires the scalar curvature to be spatially L^1 integrable, so we have to focalize on some special situations. Following Hamilton [14], a solution to equation (1) is called *non-singular* if the solution exists for all time with uniformly bounded sectional curvature.

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In our previous paper [9], the authors considered non-singular solutions to the normalized Ricci flow on compact manifolds and partially generalized Hamilton's convergence results. Remarkably, the authors found one topological obstruction for the existence of non-singular solutions on 4-manifolds, e.g., the Euler characteristic of the underlying closed 4-manifold has to be nonnegative. Moreover, if the Perelman invariant of the 4-manifold is negative, the Hitchin-Thorpe type inequality holds true, i.e.,

$$2\chi(M) \geq \frac{3}{2}|\tau(M)|$$

where $\chi(M)$ (resp. $\tau(M)$ is the Euler characteristic (resp. sinature) of the 4-manifold M (cf. [9]). Based on our methods in [9], Ishida [15] recently provided some more examples of smooth 4-manifolds which shows that the existence of long time non-singular solution really depends on the smooth structure of the underlying manifold.

In this paper we are concerned with non-singular solutions on complete manifolds with finite volume. Our first result is the following

Theorem 1.1. *Let $g(t)$ be a complete non-singular solution of equation (1) of finite volume on a noncompact n-dimensional manifold M, then either $g(t)$ collapses along a subsequence or $g(t)$ converges smoothly along a subsequence to a complete Einstein manifold with negative Einstein constant.*

Here, "collapse along a subsequence" means that $\max_{x\in M} \mathrm{inj}(x, g(t_k)) \to 0$ for certain sequence of times $t_k \to \infty$. The difference from the case of closed manifolds is the absence of shrinking Ricci solitons and Ricci flat manifolds in the limit spaces.

In dimension 4, using the above Theorem 1.1 we obtain the following

Theorem 1.2. *Let $g(t)$ be a complete non-singular solution of equation (1) of finite volume on a 4-manifold M (compact or not), then the Euler characteristic number*

$$2\chi(M) \geq \left|\frac{1}{16\pi^2}\int_M (|W_0^+|^2 - |W_0^-|^2) dv_{g(0)}\right| \geq 0,$$

where $W_0^\pm$ denotes the Weyl tensor of the initial metric $g(0)$. Moreover, $\chi(M) = 0$ (resp. $\chi(M) = 0$ and the signature $sig(M) = 0$) if and only if $g(t)$ collapses along a subsequence (resp. M is in addition compact).

When $\chi(M) \neq 0$, the Ricci flow converges to the negative Einstein manifolds on the thick part. The volume of the thin part becomes smaller and smaller, and converges to zero when the time tends to infinity:

Theorem 1.3. *Let M be as in Theorem 1.2. If $\chi(M) \neq 0$, then there exist a sequence times $t_k \to \infty$, a double sequence of points $\{p_{k,l}\}_{l=1}^N$ and domains $\{U_{k,l}\}_{l=1}^N$ with $p_{k,l} \in U_{k,l}$ satisfying the followings:*

(i). $\mathrm{dist}_{g(t_k)}(p_{k,l_1}, p_{k,l_2}) \to \infty$ *as* $k \to \infty$*, for any fixed* $l_1 \neq l_2$*;*
(ii). *for each* l*,* $(U_{k,l}, g(t_k), p_{k,l})$ *converges in the* C^∞_{loc} *sense to a complete negative Einstein manifold* $(M_{\infty,l}, g_{\infty,l}, p_{\infty,l})$ *when* $k \to \infty$*;*
(iii). $\mathrm{Vol}_{g(t_k)}(M \backslash \bigcup_{l=1}^N U_{k,l}) \to 0$ *as* $k \to \infty$*.*

Recall that a Riemannian manifold (M, g) is *asymptotic to a fibred cusp* if M is diffeomorphic to the interior of a compact manifold $\overline{M}$ whose boundary is a fibration $F \longrightarrow \partial\overline{M} \longrightarrow B$, and the metric $g \sim dr^2 + \pi^* g_B + e^{-2r} g_F$ at infinity (so the fibres collapse at infinity). The next theorem concerns the Hitchin-Thorpe type inequality for non-singular solutions on noncompact four manifolds asymptotic to a fibered cusp at infinity. We need a correction term in the Hitchin-Thorpe inequality, namely the adiabatic limit of the η-invariants of the infinity. Using the work of Dai and Wei [7] we obtain the following theorem:

Theorem 1.4. *Let* M *be as in Theorem 1.1. If* $(M, g(0))$ *is asymptotic to a fibred cusp, then the strict Hitchin-Thorpe type inequality holds:*

$$2\chi(M) > 3|\tau(M) + \frac{1}{2}\mathrm{a}\lim \eta(\partial\overline{M})| \tag{2}$$

where $\mathrm{a}\lim \eta(\partial\overline{M})$ *is the adiabatic limit of* η *invariant of the boundary.*

When $\partial\overline{M}$ has special structures, for example $\partial\overline{M}$ is a disjoint union of circle bundles over surfaces, we have more precise inequality in the above Hitchin-Thorpe type inequality (compare [7]).

Corollary 1.5. *Let* $(M, g(0))$ *be as in Theorem 1.4. If the fibration at infinity consists of circle bundles over surfaces* $S^1 \to N_i \to \Sigma_i, 1 \leq i \leq k$*, then*

$$2\chi(M) > 3|\tau(M) - \sum_i \frac{1}{3} e_i| \tag{3}$$

where e_i*,* $1 \leq i \leq k$*, are Euler numbers of the circle bundles.*

We remark that the asymptotic assumption in Theorem 1.4 (resp. Corollary 1.5) may be replaced by assuming the initial metric has bounded covering geometry and the infinity has a polarized F-structure. Moreover, using the same argument we may extend Corollary 1.5 to the case where the end of M is asymptotic to a complex hyperbolic end, i.e,

$$g \sim dr^2 + e^{-r} g_{T^2} + e^{-2r} \theta \wedge \theta$$

where θ is an invariant 1-form on the circle fibre, and $\partial\overline{M}$ is a 3-dimensional nil-manifold.

Comparing with our previous work in [9][10], it is natural to ask whether the rigidity theorem could be extended to noncompact 4-manifold. More

precisely, assume that M is a complete non-compact 4-dimensional Riemannian manifold of finite volume whose end is asymptotic to a complex hyperbolic 4-manifold, if M in addition admits a symplectic structure, can one conclude the Einstein part in Theorem 1.3 is complex hyperbolic under certain topological constraints? (compare the work of Biquard [2].)

We conclude this introduction by pointing out the main difference from the compact case dealt in [9][10]. To prove the convergence part of Theorem 1.1, a key lemma we need to verify in the non-compact case is the vanishing of the integral $\int_M \triangle R$ (cf. Lemma 3.3 below). This follows by using Shi's derivative estimate for curvatures when the sectional curvature of the solution is uniformly bounded. In the proof of Theorem 1.4 we need to estimate derivatives of the curvature operator which depends also heavily on Shi's estimation. Because of this, we really need the sectional curvature bound in the noncompact case, rather than Ricci bound or even scalar curvature bound in certain cases as in our previous works. We will get back to this point in future.

The paper is organized as follows: In Section 2, we recall one theorem about the maximal principle on noncompact manifolds; in Section 3, we prove Theorem 1.1 and then in Section 4 we prove Theorem 1.2, Theorem 1.3 and Theorem 1.4.

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2. Preliminaries

To consider the Ricci flow on noncompact manifolds, we need to use the maximal principle on noncompact manifolds. For the sake of reader's convenience let us recall the following general result, which was proved in [18], see also [5].

Theorem 2.1. [18] *Let $(M, g(t)), t \in [0, T]$, be a smooth family of complete evolving Riemannian manifolds such that $\frac{\partial}{\partial t} g_{ij} = -2\Upsilon_{ij}$. Denote $R_\star(t) = \inf_M \operatorname{tr}_{g(t)} \Upsilon$ and assume that $R_\star(g)$ is finite and integrable. Assume further that the metrics $g(t) \geq g^\star$ for a fixed complete metric $g^\star$. Then for any subsolution to the heat equation $\frac{\partial}{\partial t} u \leq \triangle_{g(t)} u$, if there is one $\alpha > 0$ and $o \in M$ such that*

$$\int_0^T \int_M \exp(-\alpha d_\star^2(o, x)) u_+^2(x, t) dv_{g(t)}(x) dt < \infty,$$

where $d_\star$ denotes the distance function of $g^\star$ and $u_+ = \max(0, u)$, then $u(0) \leq 0$ implies $u(t) \leq 0$ for all time $t \in [0, T]$.

One immediate corollary says

Corollary 2.2. [5] *Let $(M, g(t)), t \in [0, T]$, be a complete solution to the Ricci flow with uniformly bounded Ricci curvature. If u is a weak subsolution of the heat equation on $M \times [0, T]$, such that $u(0) \leq 0$ and*

$$\int_0^T \int_M \exp(-\alpha d^2_{g(0)}(o, x)) u_+^2 dv_{g(t)}(x) dt < \infty,$$

for some $\alpha > 0$, then $u \leq 0$ over $M \times [0, T]$.

3. Non-singular solutions of finite volume

We will give a proof of Theorem 1.1 in this section. Our argument relies on the following classification theorem of limit models of Type I Ricci flow, which is due to Naber [17]. By Hamilton [12], a solution to the Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2Ric(g(t)) \tag{4}$$

is called *Type I* if the curvature satisfies $\sup_M |Rm|(t) \leq \frac{C_0}{T-t}$ for some constant C_0 independent of t.

Theorem 3.1. [17] *Let $(M, g(t)), t \in [0, T)$, be a Type I solution to the Ricci flow (4) such that each metric $g(t)$ is complete, then for any given point $p \in M$ and times $t_i \to T$, the sequence of Ricci flow solutions $(M, (T - t_i)^{-1} g((T - t_i)t + t_i), p)$ will converge along a subsequence to a shrinking Ricci soliton $(M_\infty, g_\infty(t), p_\infty), t \in (-\infty, 1)$.*

In the following of this section, M stands for a noncompact manifold and $g(t), t \in [0, \infty)$, is a normalized Ricci flow solution with finite volume:

$$\frac{\partial}{\partial t} g = -2Ric + \frac{2r}{n} g, \tag{5}$$

where $r = \frac{\int R dv}{\mathrm{Vol}(g)}$ denotes the average scalar curvature as usual. Suppose $g(t)$ have uniformly bounded curvatures $|Rm|(t) \leq C$ for some constant C independent of t. Obviously the flow (6) preserves the volume and so after a scaling we may assume that $\mathrm{Vol}(g(t)) \equiv 1$ for all time.

We first establish some lemmas for the proof of the theorem. First we set $\breve{R}(g) = \inf_M R(g)$ for given metric g, the infimum of the scalar curvature on M, and let $\breve{R}(t) = \breve{R}(g(t))$. The following lemma follows from the maximal principle.

Lemma 3.2. *$\breve{R}(t)$ preserves the nonnegative property and increases whenever it is non-positive.*

Proof. Consider the evolution equation of the scalar curvature:

$$\frac{\partial}{\partial t}R = \triangle R + 2|Ric^o|^2 + \frac{2}{n}R(R-r), \tag{6}$$

where Ric^o denotes the trace free part of the Ricci curvature. We assume that $r(t)$ is a scalar function defined in advance satisfying $\breve{R}(t) \leq r(t)$ for all time. Let f be a scalar function defined by $\frac{d}{dt}f = \frac{2}{n}f(f-r)$ with initial value $f(0) = \breve{R}(0) - \epsilon$ for fixed small $\epsilon > 0$. Then $f(0) < r(0)$ and $\min(f(0), 0) \leq f(t) \leq r(t)$ for all time. Moreover, $R - f$ satisfies the evolution inequality

$$\frac{\partial}{\partial t}(R-f) \geq \triangle(R-f) + \frac{2}{n}(R+f-r)(R-f),$$

with initial condition $(R-f)(0) > 0$ over M. Suppose that $\frac{2}{n}|R+f-r| \leq \bar{C}$, then

$$\frac{\partial}{\partial t}[e^{\bar{C}t}(R-f)] \geq \triangle[e^{\bar{C}t}(R-f)]$$

whenever $R - f \leq 0$. Applying Corollary 2.2 to $e^{\bar{C}t}(R-f)$ we get that $f(t) \leq R(t)$ for all time.

If $R \geq 0$ at $t = 0$, then $f(0) \geq -\epsilon$ and $f(t) \geq -\epsilon$ for all time. So $R(t) \geq -\epsilon$ for all time and then letting ϵ tend to zero yields the nonnegativity of $R(t)$. If $\breve{R}(0) \leq 0$, then $f(t) \geq f(0)$ for all time and this gives the monotonicity of $\breve{R}$ by letting ϵ tends to zero. □

Lemma 3.3. *At each time $t \geq 1$, $\int_M \triangle R dv = 0$.*

Proof. By Shi's gradient estimate [20], see also [12], there is a constant $C_1 < \infty$ such that for any given $t \geq 1$, the estimates $|\triangle R|, |\nabla R| \leq C_1$ spatially holds at time t. Then choose a chopping $\{U_k\}$ with smooth boundaries such that $U_k \subset U_{k+1}$ and $\bigcup U_k = M$. We can also assume that $\mathrm{Vol}(\partial U_k, g|_{U_k}) \to 0$ as $k \to \infty$ because the total volume of M is finite. Then

$$\int_M \triangle R dv = \lim_{k\to\infty} \int_{U_k} \triangle R dv \leq \lim_{k\to\infty} \int_{\partial U_k} |\nabla R| = 0,$$

since $|\nabla R| \leq C_1$ and the volume $\mathrm{Vol}(\partial U_k) \to 0$. □

To prove the convergence result, we also need the following

Lemma 3.4. *If $\breve{R}(t) \leq -c < 0$ for all time, then*

$$\int_0^\infty (r - \breve{R})dt \quad < \quad \infty, \tag{7}$$

$$\int_0^\infty \int_M |R-r| dvdt < \infty, \tag{8}$$

$$\int_0^\infty \int_M |Ric^o|^2 dvdt \quad < \quad \infty. \tag{9}$$

Proof. The first estimate follows from the maximal principle. Given $\epsilon > 0$, let f be the function as defined in the proof of Lemma 3.2 which is monotone increasing and satisfies that $f(t) \leq \breve{R}(t) \leq -c$ for all time. From the evolution equation $\frac{d}{dt}f = \frac{2}{n}f(f-r)$ we obtain that

$$\int_0^\infty (r(t) - f(t))dt \leq \frac{n}{2c}(-f(0) - c).$$

So

$$\int_0^\infty (r(t) - \breve{R}(t))dt \leq \int_0^\infty (r(t) - f(t))dt < \infty.$$

The second estimate follows directly by

$$\int_0^\infty \int_M |R - r|dvdt \leq \int_0^\infty (R - \breve{R} + r - \breve{R})dvdt = \int_0^\infty 2(r - \breve{R})dt.$$

To show the third estimate, we consider the evolution

$$\begin{aligned} \frac{d}{dt}\int_M Rdv &= \int_M (\triangle R + 2|Ric^o|^2 + \frac{2-n}{n}R(R-r))dv \\ &= \int_M (2|Ric^o|^2 + \frac{2-n}{n}R(R-r))dv. \end{aligned}$$

It follows that

$$\int_0^\infty \int_M 2|Ric^o|^2 dvdt \leq \lim_{t\to\infty} |r(t) - r(0)| + \frac{n-2}{n}\int_0^\infty \int_M C|R - r|dvdt < \infty,$$

which is the desired result. □

The consequence is that the metric tends to be Einstein in the L^2 sense:

Lemma 3.5. *Suppose as in above lemma, then*

$$\lim_{t\to\infty}(r(t) - \breve{R}(t)) = 0, \tag{10}$$

$$\lim_{t\to\infty}\int_M |Ric^o|^2 dv = 0. \tag{11}$$

Proof. By above lemma, it suffice to show that

$$\frac{d}{dt}(r - \breve{R}) \leq D, \quad \text{and}$$

$$\frac{d}{dt}\int_M |Ric^o|^2 dv \leq D$$

for some uniform constant $D < \infty$. These facts follow from Shi's gradient estimate and the non-singular assumption. □

The following is the key lemma for proving Theorem 1.1:

Lemma 3.6. $\liminf_{t\to\infty} r(t) \leq 0$.

The proof of this lemma relies on a contradiction argument and we postpone it to the end of this section.

Lemma 3.7. *If* $\liminf_{t\to\infty} r(t) = 0$, *there is a sequence* $t_k \longrightarrow \infty$ *such that*

$$\lim_{k\to\infty}\int_M 2|Ric^o(t_k)|^2 dv_{g(t_k)} = 0.$$

Proof. Let $t_k \to \infty$ be a sequence with $\lim_{k\to\infty} r(t_k) = 0$. Consider the family of functions $r(t_k+t)$, $t \in [-t_k, +\infty)$. By Shi's gradient estimate [20], for any $l > 0$,

$$|\frac{d^l r(t_k+t)}{dt^l}| \le C' \sum_{0\le j\le l} |\frac{d^j}{dt^j} Ric(t_k+t)| \le C \sum_{0\le j\le 2l} |\nabla^j Rm(t_k+t)| \le \bar{C},$$

for a constant $\bar{C} > 0$ independent of t and k. By passing to a subsequence, $r(t_k+t)$ C^∞-converges to a smooth function $r_\infty(t)$ on $\mathbb{R}$, which satisfies $r_\infty(t) \ge r_\infty(0) = 0$.

Indeed, by Eq. (10) in Lemma 3.5 and $\lim_{k\to\infty} r(t_k) = 0$, we deduce that $\lim_{k\to\infty} \breve{R}(t_k) = 0$. Then by assumption that $r_\infty(0) = \min_{t\in\mathbb{R}} r_\infty(t)$,

$$\begin{aligned}
0 &= \frac{dr_\infty}{dt}(0) = \lim_{k\to\infty}\frac{dr}{dt}(t_k) \\
&= \lim_{k\to\infty}\int_M (2|Ric^o(t_k)|^2 + \frac{2-n}{n} R(t_k)(R(t_k)-r(t_k)))dv_{g(t_k)} \\
&\ge \lim_{k\to\infty}\int_M 2|Ric^o(t_k)|^2 dv_{g(t_k)} - \lim_{k\to\infty}\frac{2-n}{n} C\int_M |R(t_k)-r(t_k)| dv_{g(t_k)} \\
&\ge \lim_{k\to\infty}\int_M 2|Ric^o(t_k)|^2 dv_{g(t_k)} - \lim_{k\to\infty}\frac{2-n}{n} C\int_M (R(t_k)+r(t_k)-2\breve{R}(t_k)) dv_{g(t_k)} \\
&= \lim_{k\to\infty}\int_M 2|Ric^o(t_k)|^2 dv_{g(t_k)} - \lim_{k\to\infty}\frac{2-n}{n} 2C(r(t_k)-2\breve{R}(t_k)) \\
&= \lim_{k\to\infty}\int_M 2|Ric^o(t_k)|^2 dv_{g(t_k)}.
\end{aligned}$$

□

Now we can give a

Proof of Theorem 1.1. If $\liminf_{t\to\infty} r(t) = 0$, then we claim that for any sequence of times $t_k \to \infty$ with $\lim_{k\to\infty} r(t_k) = 0$, the corresponding metrics $g(t_k)$ collapse as $k \to \infty$. Suppose not, then there exists $\epsilon > 0$ such that passing a subsequence, for each k, there is one point $p_k \in M$ with $\mathrm{inj}(p_k, g(t_k)) \ge \epsilon > 0$. Then passing one subsequence again, by Hamilton's compactness theorem [13], $(M, g(t_k+t), p_k)$ converge to a limit "normalized"

Ricci flow solution $g_\infty(t)$ on one noncompact manifold M_∞:

$$\frac{\partial}{\partial t} g_\infty(t) = -2Ric(g_\infty(t)) + \frac{2}{n} r_\infty(t) g_\infty(t),$$

where $r_\infty(t) = \lim_{k\to\infty} r(t_k + t)$ is nonnegative satisfying $r_\infty(0) = 0$. We will show that $g_\infty(0)$ is Ricci flat, so $g_\infty(0)$ has infinite total volume (cf. [4, 21]), which contradicts with the fact $\mathrm{Vol}(M_\infty, g_\infty) \le 1$.

Indeed, by Eq. (10) in Lemma 3.5 and $\lim_{k\to\infty} r(t_k) = 0$, we deduce that $\lim_{k\to\infty} \breve{R}(t_k) = 0$. Then by Lemma 3.7

$$0 = \lim_{k\to\infty} \int_M 2|Ric^o(t_k)|^2 dv_{g(t_k)},$$

which implies that $g_\infty(0)$ is Einstein. $\breve{R}(t_k) \to 0$ yields that $g_\infty(0)$ has nonnegative scalar curvature. Thus $g_\infty(0)$ must be Ricci flat since the manifold M_∞ is noncompact.

On the other hand, if $\liminf_{t\to\infty} r(t) = -c < 0$, then Eq. (10) implies that $\lim_{t\to\infty} \breve{R}(t) = -c < 0$. If $g(t)$ do not collapse along a sequence $t_k \to \infty$, then (M, g_{t_k}) converge subsequently to a limit by Hamilton's compactness theorem [13]. By Eq. (11), the limit must be negative Einstein. $\square$

At last we give a

Proof of Lemma 3.6. Argue by contradiction. By contraries, there is $\delta > 0$ such that $r(t) \ge \delta$ for all time. As showed in [9], in this situation, the corresponding unnormalized Ricci flow becomes singular in finite time and the total volume tends to zero as the solution approaches the singular time. We can claim more on the volume decay rate:

Claim 3.8. *Let $\tilde{g}(\tilde{t}) = \psi(t)g(t), \tilde{t} \in [0, \widetilde{T})$, be the corresponding Ricci flow solution. Then there is $C_2 < \infty$ such that*

$$C_2^{-1}(\widetilde{T} - \tilde{t})^{n/2} \le \mathrm{Vol}(\tilde{g}(\tilde{t})) \le C_2(\widetilde{T} - \tilde{t})^{n/2}.$$

Proof. Comparing the evolution of Ricci flow

$$\frac{\partial}{\partial \tilde{t}} \tilde{g} = -2Ric(\tilde{g}), \tag{12}$$

with normalized Ricci flow equation (5), we obtain the identities

$$\begin{aligned} \frac{\partial \tilde{t}}{\partial t} &= \psi(t) \\ 0 &= \frac{\partial}{\partial t}(\ln \psi) + \frac{2}{n} r(t). \end{aligned}$$

So the scaling function $\psi(t) = \mathrm{Vol}(\tilde{g}(t))^{2/n}$ is given by the integration $\psi(t) = \exp(-\int_0^t \frac{2}{n} r(s)ds)$. And

$$\tilde{t} = \int_0^{\tilde{t}} d\tilde{t} = \int_0^t \psi(s)ds = \int_0^t \exp(-\int_0^s \frac{2}{n} r(u)du)ds,$$

in particular $\widetilde{T} = \int_0^\infty \exp(-\int_0^s \frac{2}{n} r(u)du)ds$. Now we can compute

$$\begin{aligned}(\widetilde{T} - \tilde{t})\,\mathrm{Vol}(\tilde{g}(\tilde{t}))^{-\frac{2}{n}} &= \psi(t)^{-1}(\widetilde{T} - \tilde{t}) \\ &= \exp(\int_0^t \frac{2}{n} r(u)du) \cdot \int_t^\infty \exp(-\int_0^s \frac{2}{n} r(u)du)ds \\ &= \int_t^\infty \exp(-\int_t^s \frac{2}{n} r(u)du)ds,\end{aligned}$$

which is bounded from above by $\frac{n}{2\delta}$ and below $\frac{n}{2C}$. Then letting $C_2 = \max(\frac{n}{2\delta}, \frac{2C}{n})$ we get he desired estimate. $\square$

Next we claim that

Claim 3.9. *The unnormalized Ricci flow* $(M, \tilde{g}(\tilde{t})), \tilde{t} \in [0, \widetilde{T})$, *is of Type I.*

Proof. By assumption, the average scalar curvature $r(t)$ is comparable with $\sup |Rm|(t)$ for each time and so it suffice to check that the quantity

$$\tilde{r}(\tilde{t})(\widetilde{T} - \tilde{t}) \;=\; r(t)\psi(t)^{-1}(\widetilde{T} - \tilde{t})$$

has a uniform upper bound for all $\tilde{t} < \widetilde{T}$, which follows directly from the arguments in the proof of Claim 3.8. $\square$

The Claim 3.8 shows that $g(t) = \alpha(\tilde{t}) \cdot (\widetilde{T} - \tilde{t})^{-1}\tilde{g}(\tilde{t})$ for certain family of bounded constants $C_2^{-1} \le \alpha(\tilde{t}) \le C_2$. Then applying Theorem 3.1, $(M, g(t))$ converge along a subsequence to a noncompact shrinking Ricci soliton, say (M_∞, g_∞), with bounded curvature and finite volume. This is a contradiction because the volume $\mathrm{Vol}(M_\infty, g_\infty)$ must be infinite. In fact, from Carrillo and Ni's work [6], Perelman's μ functional $\mu(g_\infty, 1)$ (see [6] for a definition) is bounded below. Following Perelman's proof of no local collapsing of finite time Ricci flow, cf. [19, §4] or [16, §13], one can prove that the volume of any unit metric ball in M_∞ has a uniform lower bound (notice here the soliton g_∞ has bounded curvature and so Bishop-Gromov volume comparison theorem works), and so $\mathrm{Vol}(g_\infty) = \infty$. This finishes the proof of the Lemma 3.6. $\square$

4. Non-singular solutions on 4-manifolds with finite volume

In this section, we restrict ourselves on 4-dimensional case and prove some similar result as in the compact case (cf. [9]).

Proof of Theorem 1.2. Consider Gauss-Bonnet-Chern formula for a Riemannian manifold of finite volume and bounded curvature:

$$\chi(M)=\frac{1}{8\pi^2}\int_M(\frac{R^2}{24}+|W|^2-\frac{1}{2}|Ric^o|^2)dv. \tag{13}$$

First of all, if $g(t)$ collapses along a subsequence, then by Cheeger-Gromov's F-structure theory [3], M admits local non-trivial tori actions and so $\chi(M)=0$. Actually we can say more about the relation between $\chi(M)$ and the collapsing in our situation:

Claim 4.1. *The following three conditions are equivalent:*

(1) $\chi(M)=0$; (2) $\liminf_{t\to\infty}\breve{R}(t)=0$;
(3) $g(t)$ *collapses along a subsequence.*
(4) $\chi(M)=sig(M)=0$ *if M is in addition compact.*

Proof of the Claim. By above observation and Claim 3.8, it suffice to show that (1) implies (2) and (3) implies (4). Suppose not, then by Claim 3.8 again, $\lim_{t\to\infty}\breve{R}(t)=-c$ for some positive constant c. Then by Lemma 3.5, $r(t)\to -c$ and $\int_M|Ric^o|^2dv\to 0$ as $t\to\infty$. Applying the Gauss-Bonnet-Chern formula,

$$\begin{aligned}
\chi(M) &= \frac{1}{8\pi^2}\int_M(\frac{R^2}{24}+|W|^2-\frac{1}{2}|Ric^o|^2)dv\\
&\geq \frac{1}{192\pi^2}\mathrm{Vol}(g(t))r(t)^2-\frac{1}{16\pi^2}\int_M|Ric^o|^2dv\\
&\to \frac{c^2}{192\pi^2}\mathrm{Vol}(g(0)),
\end{aligned}$$

as $k\to\infty$, which contradicts with the assumption $\chi(M)=0$. This finishes the proof of the claim.

To see (3) implies (4) when M is a compact. We claim that Lemma 3.6 still holds for this case, i.e. $\liminf_{t\to\infty}r(t)\leq 0$. Otherwise, there is a sequence $t_j\longrightarrow\infty$ such that $g(t_j+t)$ collapses, and $r(t_j+t)\geq\delta$ for a constant $\delta>0$ independent of j and t. However, by the same arguments in the proof of Lemma 3.6, a subsequence of $g(t_j+t)$ converges to a shrinking soliton (M_∞,g_∞), which contradicts to (3).

By the proof of Lemma 3.7 we get a sequence $t_k\longrightarrow\infty$ such that

$$\lim_{k\to\infty}\int_M 2|Ric^o(t_k)|^2dv_{g(t_k)}=0,$$

and, hence (4) follows from below:

$$2\chi(M)-3|\tau(M)|\geq \lim_{k\to\infty}\frac{1}{4\pi^2}\int_M(\frac{R^2(t_k)}{24}-\frac{1}{2}|Ric^o(t_k)|^2)dv_{g(t_k)}\geq 0.$$

□

Now we prove that $\chi(M)\geq 0$. Indeed, if $\chi(M)\neq 0$, then by above claim and Lemma 3.6, $\breve{R}(t)\to -c$ for some $c>0$ and then the previous arguments imply that $\chi(M)\geq \frac{c^2}{192\pi^2}\operatorname{Vol}(g(0))>0$. The desired result follows.

Furthermore, we have

$$2\chi(M)-|\frac{1}{16\pi^2}\int_M(|W_t^+|^2-|W_t^-|^2)dv_{g(t)}|\geq \frac{1}{4\pi^2}\int_M(\frac{R^2}{24}-\frac{1}{2}|Ric^o|^2)dv.$$

By Lemma 3.5 and 3.7, there is a sequence $t_k\longrightarrow\infty$ such that

$$\lim_{t_k\longrightarrow\infty}\int_M|Ric^o_{t_k}|^2dv_{t_k}=0,$$

and, hence,

$$2\chi(M)\geq \lim_{t_k\longrightarrow\infty}|\frac{1}{16\pi^2}\int_M(|W_{t_k}^+|^2-|W_{t_k}^-|^2)dv_{g(t_k)}|.$$

Since $(|W_t^+|^2-|W_t^-|^2)dv_{g(t)}$ is the first Pontryagin form up to a positive multiplication by Chern-Weil theory, we obtain the desired result from the following claim. By now the desired result follows from the following claim.

□

Claim 4.2. *For any characteristic polynomial P on M^4, we have*

$$\int_M P(\Omega_t)=\int_M P(\Omega_0),\quad \forall t\geq 0. \tag{14}$$

Proof of the Claim. The non-singular assumption says that $|Rm(t)|\leq C$ for some constant $C<\infty$ independent of t. Then by Shi's first gradient estimate [20], there exists another constant $C_4=C_4(n,C)$ such that $|\nabla Rm(t)|\leq C_4(1+t^{-1/2})$ for all $t>0$. Denote by Ω_t the curvature operators of metrics $g(t)$.

Given $\rho>0$, denote by M_ρ the set of points whose radial coordinate at infinity with respect to the metric $g(0)$ are less that ρ. Then by the evolving equation of the volume form, we have

$$|\frac{d}{dt}\ln\operatorname{Vol}_{g(t)}(M\backslash M_\rho)|=|\operatorname{Vol}^{-1}_{g(t)}(M\backslash M_\rho)\int_{M\backslash M_\rho}(r-R)dv_{g(t)}|\leq 2C;$$

$$\begin{aligned}|\frac{d}{dt}\ln\operatorname{Vol}_{g(t)}(\partial\overline{M}_\rho)| &= |\operatorname{Vol}^{-1}_{g(t)}(\partial\overline{M}_\rho)\int_{\partial\overline{M}_\rho}(\frac{3}{4}r-R+Ric(\nu,\nu))dv_{g(t)}|\\ &\leq 3C,\end{aligned}$$

where ν is the normal vector fields on $\partial\overline{M}_\rho$. For fixed time $t_0 > 0$,

$$\mathrm{Vol}_{g(t_0)}(M\backslash M_\rho) \leq e^{2Ct_0}\,\mathrm{Vol}_{g(0)}(M\backslash M_\rho) \to 0$$

and

$$\mathrm{Vol}_{g(t_0)}(\partial\overline{M}_\rho) \leq e^{3Ct_0}\,\mathrm{Vol}_{g(0)}(\partial\overline{M}_\rho) \to 0$$

as $\rho \to \infty$, since the total volume of $g(0)$ is finite and $\mathrm{Vol}_{g(0)}(\partial\overline{M}_\rho) \to 0$ exponentially as $\rho \to \infty$. Combing with Eq. (16) and (17), and using the curvature bound $|Rm|(t) \leq C$, it follows that

$$\begin{aligned}
\int_M P(\Omega_{t_0}) &= \lim_{\rho\to\infty}\int_{M_\rho} P(\Omega_{t_0}) \\
&= \lim_{\rho\to}\int_{M_\rho} P(\Omega_0) + \lim_{\rho\to\infty}\int_{M_\rho}(P(\Omega_{t_0}) - P(\Omega_0)) \\
&= \int_M P(\Omega_0) + \lim_{\rho\to\infty}\int_{M_\rho} 2d\int_0^{t_0} P(\dot{\omega}_t, \Omega_t)dt \\
&= \int_M P(\Omega_0) + \lim_{\rho\to\infty}\int_{\partial\overline{M}_\rho}\int_0^{t_0} 2P(\dot{\omega}_t, \Omega_t)dt, \qquad (15)
\end{aligned}$$

since $deg(P) = 2$. In local coordinate $(x^1, \cdots, x^4)$, let

$$\Gamma_{ij}^k(t) = \frac{1}{2}g^{kl}(t)(\frac{\partial g_{il}(t)}{\partial x^j} + \frac{\partial g_{jl}(t)}{\partial x^i} - \frac{\partial g_{ij}(t)}{\partial x^l})$$

be the Christoffel symbols of the Levi-Civita connection at time t. Then we can rewrite the connection one form by $\omega_t(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \Gamma_{ij}^k(t)\frac{\partial}{\partial x^k}$ and so by the evolution of the Ricci flow equation (1):

$$\dot{\omega}_t(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = g^{kl}(t)(-\nabla_i R_{jl} - \nabla_j R_{il} + \nabla_l R_{ij})\frac{\partial}{\partial x^k},$$

where ∇ denotes the Levi-Civita connection of $g(t)$. Thus

$$|\dot{\omega}_t| \leq 3C_4(1 + t^{-1/2}).$$

Consequently using the assumption that $|\Omega_t| = |Rm| \leq C$,

$$\begin{aligned}
|\int_{\partial\overline{M}_\rho}\int_0^{t_0} P(\dot{\omega}_t, \Omega_t)dt| &\leq C''\int_{\partial\overline{M}_\rho}\int_0^{t_0}|\dot{\omega}_t|\cdot|\Omega_t|dv_{g(t)}dt \\
&\leq 3C''C_4C\int_0^{t_0}(1 + t^{-1/2})\,\mathrm{Vol}_{g(t)}(\partial\overline{M}_\rho)dt \\
&\leq 3C''C_4C\,\mathrm{Vol}_{g(0)}(\partial\overline{M}_\rho)\cdot\int_0^{t_0} e^{3Ct}(1 + t^{-1/2})dt \\
&\leq 3C''C_4C(2t_0^{1/2} + t_0)e^{3Ct_0}\cdot\mathrm{Vol}_{g(0)}(\partial\overline{M}_\rho) \\
&\to 0
\end{aligned}$$

as $\rho \to \infty$, where C'' is a constant depending on the coefficients of the polynomial P. Substituting into (15), the desired result follows. □

Proof of Theorem 1.3. The proof depends on explicit estimate by using Gauss-Bonnet-Chern formula and Cheeger-Gromov's collapsing theory (with bounded curvature). We only give a sketch proof here since its proof is totally the same as in the compact case. By Claim 4.1 and Lemma 3.6, we may assume that $\breve{R}(t) \to -c < 0$ as $t \to \infty$.

Choose $t_k \to \infty$. Fix one small constant $\varepsilon > 0$ such that $M_{k,\epsilon} =: \{x \in M | \operatorname{Vol}_{g(t_k)}(B_{g(t_k)}(x, 1)) < \varepsilon\}$ admits an F-structure of positive rank, cf. [3]. Passing a subsequence, there is a uniform constant $N \leq \frac{1}{\varepsilon}$ satisfying that for each k, we can find a maximal set of points $\{p_{k,l}\}_{l=1}^{N_1} \subset M \backslash M_{k,\varepsilon}$ such that as $k \to \infty$,

$$3\rho_k = \min\{\operatorname{dist}_{g(t_k)}(p_{k,l_1}, p_{k,l_2}) | l_1 \neq l_2\} \to \infty,$$

$$\operatorname{Vol}_{g(t_k)}\Big(\bigcup_{l=1}^{N} \partial B_{g(t_k)}(p_{k,l}, \rho_k)\Big) \to 0,$$

and the second fundamental forms Π of $\partial B_{g(t_k)}(p_{k,l}, \rho_k))$ are uniformly bounded. Then applying Hamilton's compactness theorem for Ricci flow, cf. [12], by Theorem 1.1, passing a subsequence again we get that for each l, $(B_{g(t_k)}(p_{k,l}, \rho_k), g(t_k), p_{k,l})$ converges smoothly to a complete negative Einstein manifold $(M_{\infty,l}, g_{\infty,l}, p_{\infty,l})$.

We next show that $\operatorname{Vol}_{g(t_k)}(M \backslash \bigcup_{l=1}^{N} B_{g(t_k)}(p_{k,l}, \rho_k)) \to 0$ as $k \to \infty$. By the choice of the points $\{p_{k,l}\}_{l=1}^{N}$, we know that

$$M \backslash M_{k,\varepsilon} \subset \{x \in M | \operatorname{dist}(x, \{p_{k,l}\}_{l=1}^{N}) \leq C_3\}$$

for some constant C_3 independent of k, when k is large enough. It concludes that $M \backslash \bigcup_{l=1}^{N} B_{g(t_k)}(p_{k,l}, \rho_k) \subset M_{k,\varepsilon}$ for large k. So $\chi(M \backslash \bigcup_{l=1}^{N} B_{g(t_k)}(p_{k,l}, \rho_k)) =$

0 and by Gauss-Bonnet-Chern formula, using the assumption $|R| \leq C$,

$$
\begin{aligned}
0 &= \chi(M\backslash\bigcup_{l=1}^{N} B_{g(t_k)}(p_{k,l},\rho_k)) \\
&= \frac{1}{8\pi^2}\int_{M\backslash\bigcup_{l=1}^{N} B_{g(t_k)}(p_{k,l},\rho_k)} (\frac{1}{24}R^2 + |W|^2 - \frac{1}{2}|Ric^o|^2)dv \\
&\quad + \int_{\bigcup_{l=1}^{N}\partial B_{g(t_k)}(p_{k,l},\rho_k)} P(\Pi)dv \\
&\geq \frac{1}{8\pi^2}\int_{M\backslash\bigcup_{l=1}^{N} B_{g(t_k)}(p_{k,l},\rho_k)} (\frac{1}{24}\breve{R}^2 + \frac{1}{24}(R+\breve{R})(R-\breve{R}) - \frac{1}{2}|Ric^o|^2)dv \\
&\quad + \int_{\bigcup_{l=1}^{N}\partial B_{g(t_k)}(p_{k,l},\rho_k)} P(\Pi)dv \\
&\geq \frac{c^2}{192\pi^2}\mathrm{Vol}_{g(t_k)}(M\backslash\bigcup_{l=1}^{N} B_{g(t_k)}(p_{k,l},\rho_k)) \\
&\quad -\frac{1}{8\pi^2}\int_{M\backslash\bigcup_{l=1}^{N} B_{g(t_k)}(p_{k,l},\rho_k)} (\frac{C}{12}(R-\breve{R}) + \frac{1}{2}|Ric^o|^2)dv \\
&\quad + \int_{\bigcup_{l=1}^{N}\partial B_{g(t_k)}(p_{k,l},\rho_k)} P(\Pi)dv,
\end{aligned}
$$

which implies that $\mathrm{Vol}_{g(t_k)}(M\backslash\bigcup_{l=1}^{N} B_{g(t_k)}(p_{k,l},\rho_k)) \to 0$ as $k \to \infty$, since the last two terms tend to zero by Lemma 3.5 and the assumptions described above. Here $P(\Pi)$ denotes some polynomial of the second fundamental form Π. In the last inequality we used the monotonicity of $\breve{R}$ which implies that $\breve{R}^2 \geq c^2$ for all time. This finishes the proof of the theorem. □

Remark 4.3. By Theorem 1.3 it is easy to see that, if $\chi(M) \neq 0$, the number of Einstein pieces $N \geq 1$. Clearly, every piece contributes at least 1 to the Euler number $\chi(M)$ and so, $N \leq \chi(M)$.

Before proving Theorem 1.4, let's first recall some groundwork on the Chern-Weil theory and Chern-Simons correction term, cf. [22]. Let (N, h) be an oriented Riemannian $2n$-manifold. By Chern-Weil theory, any $SO(2n)$ invariant polynomial of degree n, say P, defines a characteristic form $P(\Omega)$, where $\Omega \in \Lambda^2 N \otimes \Lambda^2 N$ denotes the curvature operator. If we have a smooth family of metrics $h_t, t \in [t_1, t_2]$, then the Chern-Simons form Q_P, associated to P, is defined by the equation:

$$Q_P(h_{t_2}, h_{t_1}) = n\int_{t_1}^{t_2} P(\dot{\omega}_t, \Omega_t, \cdots, \Omega_t)dt \tag{16}$$

which determines the nice correction term

$$P(\Omega_{t_2}) - P(\Omega_{t_1}) = dQ_P(h_{t_2}, h_{t_1}), \tag{17}$$

where ω_t and Ω_t denote the connection one form and the curvature form of the metric h_t respectively. In our consideration, P will be the Pfaffian Pf or L-polynomial characteristic form.

Using Claim 4.2, and the Atiyah-Patodi-Singer index formula [1] on manifolds with boundary, Dai and Wei proved in [7] the following theorem for manifolds with fibred cuspidal infinity:

Theorem 4.4. [7] *Let (N, h) be a complete Riemannian 4-manifold which is asymptotic to a fibred cusp metric at infinity, then the Euler number $\chi(N)$ and signature $\tau(N)$ are given by*

$$\chi(N) = \int_N Pf(\frac{\Omega}{2\pi}); \tag{18}$$

$$\tau(N) = \int_N L(\frac{\Omega}{2\pi}) - \frac{1}{2}\mathrm{a}\lim \eta(\partial N), \tag{19}$$

where $\mathrm{a}\lim \eta(\partial N)$ *denotes the adiabatic limit of* $\eta(\partial N)$.

Here the adiabatic limit $\mathrm{a}\lim \eta(\partial N)$ is a topological invariant of the 3-manifold ∂N. Now we are ready to give a proof of Theorem 1.4:

Proof of Theorem 1.4. Let $g(t)$ be a non-singular solution on noncompact 4-manifold M such that $g(0)$ has asymptotical fibred cusps at infinity. Topologically M is the interior of a manifold $\overline{M}$ whose boundary admits a fibration structure

$$F \longrightarrow \partial\overline{M} \xrightarrow{\pi} B$$

for closed manifolds B, F and geometrically the metric $g(0)$ has the form

$$g(0) \sim dr^2 + \pi^* g_B + e^{-2r} g_F$$

at infinity, where g_B is a metric on B and $g_F = g_F(b)$, $b \in B$, is a family of metrics on F.

Using Dai and Wei's Theorem 4.4, we obtain that at any time t,

$$\chi(M) = \int_M Pf(\Omega_0) = \int_M Pf(\Omega_t); \tag{20}$$

$$\tau(M) = \int_M L(\Omega_0) - \frac{1}{2}\mathrm{a}\lim \eta(\partial\overline{M}) = \int_M L(\Omega_t) - \frac{1}{2}\mathrm{a}\lim \eta(\partial\overline{M}). \tag{21}$$

More precisely, at any time t, we have the Gauss-Bonnet-Chern formula

$$\chi(M) = \frac{1}{8\pi^2}\int_M (\frac{R^2}{24} + \frac{1}{4}|W|^2 - \frac{1}{2}|Ric^o|^2) dv_{g(t)} \tag{22}$$

and the generalized Hirzebruch signature formula

$$\tau(M) = \frac{1}{48\pi^2}\int_M (|W^+|^2 - |W^-|^2) dv_{g(t)} - \frac{1}{2}\mathrm{a}\lim \eta(\partial\overline{M}). \tag{23}$$

It follows that at any time t,

$$2\chi(M) - 2|\tau(M) + \frac{1}{2}\mathrm{a}\lim \eta(\partial\overline{M})| \geq \frac{1}{4\pi^2}\int_M (\frac{R^2}{24} - \frac{1}{2}|Ric^o|^2) dv. \tag{24}$$

Then combining Lemma 3.6, Claim 4.1 and Lemma 3.5 derives the desired strict Hitchin-Thorpe type inequality (11) by letting $t \to \infty$. □

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